$[(2\times7)+1][(4\times7)/(4^2+7^2)]=13\times15\times\frac{28}{65}=3\times4\times7=84$; $[(2\times26)-1][(2\times26+1]$ $[(15\times26)/(15^2+26^2)]=51\times53\times\frac{390}{901}=3\times15\times26=1170$; $[(2\times97)-1][(2\times97+1]$ $[(56\times97)/(56^2+97^2)]=193\times195\times\frac{5432}{12545}=3\times56\times97=16296$; $[(2\times362)-1][(2\times362)+1]$ $[(209\times362)/(209^2+362^2)]=723\times725\times\frac{75658}{174725}=3\times209\times362=226974$; etc.

Here it should be noticed that in canceling both sides the denominator of the half-sine disappears. and three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ brings the area to light ; also observe, since sides and denominator fall out of view, and factor 3 stands constant, the area must be determined by the numerator of these half-sines, and *this* series may be continued by use of Magic $M=14$; $14\times2=28$; $(14\times28)-2=390$; $(14\times390)-28=5432$; $(14\times5432)-390=75658$; etc.

Also solved by *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

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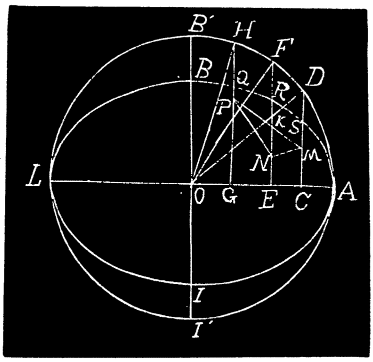
60. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral ?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

We will solve this problem for the quadrant, the semi-ellipse, and the whole ellipse.

Let $ABLI$ be the ellipse, and $AB'LI'$ the circumscribing circle ; M, N, P the three random points ; through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP at K. The triangle will pass through all the possible variations by considering only those relative positions of the points in which CD lies to the right of GH, and EF between CD and GH.



If the fourth point falls anywhere on the triangle formed by joining the points M, N, P, the quadrilateral thus formed will be reentrant.

Let $OA=a$, $OB=b$, $GP=x$, $CM=y$, $EN=z$, $GQ=x'$, $CS=y'$, $ER=z'$, $EK=z''$, $\angle GOH=\theta$, $\angle COD=\varphi$, $\angle EOF=\psi$.

Then we have $x'=b\sin\theta$, $y'=b\sin\varphi$, $z'=b\sin\psi$, $v=1/(\cos\varphi-\cos\theta)$, $z''=v[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]$.

Area $MNP=\frac{1}{2}a[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\varphi)]=u$, when $z<z''$. Area $MNP=\frac{1}{2}a[x(\cos\psi-\cos\varphi)+y(\cos\theta-\cos\psi)+z(\cos\varphi-\cos\theta)]=u_1$, when

$z>z''$.. An element of surface at M is $a\sin\varphi d\varphi dy$, at N it is $a\sin\psi d\psi dz$, at P it is $a\sin\theta d\theta dx$.

The limits of θ are (for quadrant) 0 and $\frac{1}{2}\pi$; of φ, 0 and θ ; of ψ, φ and θ ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is,

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'}a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdydz}$$

$$=\frac{384}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)\sin\theta\sin\varphi\sin\psi d\theta d\varphi d\psi dxdy$$

$$=\frac{96a}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y}\{[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]^2+[x(\cos\varphi-\cos\psi)$$

$$+y(\cos\psi-\cos\theta)+b\sin\psi(\cos\theta-\cos\psi)]^2\}\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dxdy$$

$$=\frac{32a}{\pi^3b^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}[6x^2\sin\varphi(\cos\varphi-\cos\psi)^2+6bx\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi$$

$$-\cos\theta)+6bx\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)+2b^2\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3b^2\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2+3b^2\sin^2\varphi\sin\psi(\cos\theta-\cos\varphi)(\cos\psi-\cos\theta)]$$

$$\times\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dx.$$

$$\Delta=\frac{32ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}[2\sin^3\theta\sin\varphi(\cos\varphi-\cos\psi)^2+2\sin\theta\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3\sin^2\theta\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi-\cos\theta)+3\sin\theta\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2$$

$$+3\sin^2\theta\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)$$

$$+3\sin\theta\sin^2\varphi\sin\psi(\cos\psi-\cos\theta)(\cos\theta-\cos\varphi)]\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi$$

$$=\frac{16ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}[4\sin^2\theta\cos^2\varphi+4\sin^2\varphi\cos^2\varphi+4\sin^2\theta\cos^2\theta+4\sin^2\varphi\cos^2\theta$$

$$+\sin^2\theta\cos\theta\cos\varphi+\sin^2\varphi\cos\varphi\cos\theta-6\sin\theta\cos\theta\sin\varphi\cos\varphi$$

$$+6\cos^3\theta\cos\varphi+6\cos\theta\cos^3\varphi+12+6\cos^2\theta+6\cos^2\varphi-36\cos\theta\cos\varphi$$

$$-12\sin\theta\sin\varphi-9(\theta-\phi)\sin\theta\cos\varphi+9(\theta-\phi)\sin\varphi\cos\theta]\sin^2\theta\sin^2\varphi d\theta d\varphi$$

$$=\frac{8ab}{9\pi^3}\int_0^{\frac{1}{2}\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta$$

$$=\frac{ab}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

For the semi-ellipse above the major axis, the limits of θ are 0 and π, and those of the other variables the same as above. The number of ways the three points can be taken in the semi-ellipse is eight times the number of ways in a quadrant, and hence we get

$$\triangle_1=\frac{ab}{9\pi^3}\int_0^{\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=\frac{ab}{\pi}\left(\frac{35}{24}-\frac{32}{3\pi^2}\right).$$

For the limits of θ are 0 and 2π, and the points can be taken eight times the number of ways in semi-ellipse. Hence

$$\triangle_2=\frac{ab}{72\pi^3}\int_0^{2\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=35ab/48\pi.$$

Let C, C_1, C_2 be the respective chances required.

$$C=\frac{4\triangle}{\pi ab}=\frac{4}{\pi^2}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right);\ C_1=\frac{4\triangle_1}{\pi ab}=\left(\frac{35}{42}-\frac{32}{3\pi^2}\right);$$

$$C_2=\frac{4\triangle_2}{\pi ab}=\frac{35}{12\pi^2}.$$